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Attitude Feedback Control: Unconstrained and Nonholonomic Constrained Cases

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The problem is considered of asymptotically driving, the attitude of a spacecraft via feedback control. First, a simple feedback control structure based on the measure of the equivalent angle axis is determined for a fully actuated system. Then, the same reasoning is used to find the solution for a spacecraft operating in failure mode, that is, whenever the angular velocity vector is constrained to lie in one of the spacecraft's coordinate planes. The basic aim is to show that with a simple choice for the attitude parameterization, a simple and effective structure for the feedback control law can be obtained.

Introduction

THE problem of asymptotically driving the attitude of a spacecraft via feedback regulation is a standard one and can be addressed via a number of attitude representations that give rise to different and effective control solutions. It is of greater interest to understand if any of the feasible solutions under nominal operation conditions could be adopted (or extended) in case of failure. In particular, we are interested in the case of failures that limit the spacecraft's attitude maneuvering capabilities and which constrain (in a purely cinematic framework) the controllable vehicle angular velocities that lie in a given plane fixed with respect to any spacecraft's reference frame.

The general aspects characterizing the constrained problem have received a certain amount of attention in the past few years because of their relevance in the area of nonlinear control of nonholonomic systems (see Ref. 1). A complete survey on this subject may be found in Ref. 2.

Within this framework, the problem of stabilizing the attitude of a spacecraft subject to nonholonomic constraints has been investigated in depth. In particular, in Ref. 3, it has been proven that such a class of systems, even if controllable, cannot be stabilized to an arbitrary equilibrium point if 1) a nonzero initial angular velocity exists directed along the nonactuated axis and 2) smooth state feedback control laws are searched for. More precisely, if such an initial angular velocity exists, the system can be stabilized to any arbitrary equilibrium point only if discontinuous state feedback laws are allowed. In the case where smooth state-feedback laws are adopted and in presence initial velocities as those just indicated, it has been shown that the system is anyway stabilizable with respect to an attractor, represented by a circular motion about the desired equilibrium point.

From the results of Ref. 3, only initial angular velocities compatible with the nonholonomic constraints allow attainment of any desired equilibrium attitude using smooth state-feedback control laws. For this particular condition, a nonlinear stabilizing smooth feedback law based on the use of a three-dimensional Euler-angles parameterization for the spacecraft attitude has been proposed. More recently, a new and very specific kind of parameterization for representing the spacecraft attitude was introduced. This approach allowed simplifying the structure of previously proposed smooth control laws, including those presented in Ref. 3 and in many of the references listed in Refs. 4 and 5, while also enhancing the overall control performances.

In the present work we shall consider the well-known equivalent angle-axis (Euler axis plus angle of rotation) parameterization. Many published works rely on such a representation (see Refs. 6–8). However, in the mentioned work, the control laws are found in the unconstrained case using quaternions to define the attitude of the body. It is the authors' opinion that for the present work simpler and more intuitive results can be obtained using the properties of the equivalent angle-axis formulation directly. In fact, for such a representation, it is possible to design a very simple smooth state-feedback control strategy for the fully actuated case ensuring the asymptotic stability of any desired equilibrium point. Then, it will be shown that under possible failure conditions (assuming zero initial spin rate along the unactuated axis) by using the same representation the original control law for the nominal system can be generalized leading to a time-invariant and almost everywhere smooth feedback control law. Such a closed-loop control strategy allows the spacecraft attitude to be asymptotically driven toward any desired equilibrium point.

The paper is organized as follows. In the next section the kinematic equations of a reference frame whose attitude is parameterized via the equivalent angle-axis formulation are derived. In the third section, proposed methods for the feedback control design either in the unconstrained and the constrained cases are presented. In the fourth section, the problem of a preliminary maneuvering needed for specific initial conditions is stated, and a possible solution algorithm is proposed.

Kinematic Equations

Consider the spacecraft fixed frame $\langle s\rangle$, whose attitude and angular velocity vector with respect to an assigned reference frame $\langle o\rangle$ are, at each time instant, represented by the orthonormal rotation matrix R and the geometric vector $\boldsymbol{\omega}$, respectively. Then, denoting ${}^s\boldsymbol{\omega}$ and ${}^o\boldsymbol{\omega}$ as the algebraic vectors corresponding to the projections of $\boldsymbol{\omega}$ on frames $\langle s\rangle$ and $\langle o\rangle$, respectively (thus satisfying the conditions ${}^o\boldsymbol{\omega} = R\,{}^s\boldsymbol{\omega}$ and ${}^s\boldsymbol{\omega} = R^T\,{}^o\boldsymbol{\omega}$) we may recall the well-known strapdown differential equation

$$\dot{R} = R[{}^s\boldsymbol{\omega}\wedge] \tag{1}$$

$$\dot{R} = [{}^o\boldsymbol{\omega}\wedge]R \tag{2}$$

that relate the time derivative of R to ${}^s\boldsymbol{\omega}$ and/or ${}^o\boldsymbol{\omega}$, via the use of their skew-symmetric matrix form for representing the projection of vector cross-product operations.

Consider an initial condition such that $\langle s\rangle$ and $\langle o\rangle$ are parallel, that is, $R = I$. Then, the solution of one of the first two equations shows that any attitude R can be reached via a single rotation of an angle $\theta \in [0, \pi]$ along a unit vector $\boldsymbol{v}$, where θ and $\boldsymbol{v}$ satisfy

$$R = e^{[\boldsymbol{v}\wedge]\theta} \tag{3}$$

The vector

$$\boldsymbol{\delta} \triangleq \boldsymbol{v}\theta \tag{4}$$

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is the so-called equivalent angle-axis geometric vector that admits the same projection on both $\langle s \rangle$ and $\langle o \rangle$, that is,

$$^{s}\boldsymbol{\delta} = {}^{o}\boldsymbol{\delta} \triangleq \boldsymbol{\delta} = \boldsymbol{v}\theta \tag{5}$$

which implies the fulfillment of the following eigenvector conditions:

$$(I - R)\boldsymbol{\delta} = 0, \qquad (I - R)\boldsymbol{v} = 0 \tag{6}$$

Note that $\theta \in [0, \pi]$ implies that we are considering the minimum (positive) angle allowed to be used to reach attitude R.

Without loss of generality, throughout the following development we shall always adopt the point of view of an observer located on the spacecraft (fixed) frame. Thus, for further developments we shall always consider all of the needed vectors as projected on frame $\langle s \rangle$. In particular, for ease of notation, hereinafter we shall use the simple form $\boldsymbol{\omega}$ instead of $^{s}\boldsymbol{\omega}$.

Hence, the time evolution of the attitude of $\langle s \rangle$ can be represented by the time evolution of the vector $\boldsymbol{\delta}$. To find the desired representation, let us first assume $\theta \in (0, \pi)$ and then boundary cases $\theta = 0$ and $\theta = 0$ later.

Case 1: $\theta \in (0, \pi)$

By differentiating both sides of Eq. (4) with respect to time, we obtain

$$\dot{\boldsymbol{\delta}} = \boldsymbol{v}\dot{\theta} + \dot{\boldsymbol{v}}\theta \tag{7}$$

Then, by considering that (see, for instance, Ref. 9 for the proof)

$$\dot{\theta} = \boldsymbol{v}^T\boldsymbol{\omega} \tag{8}$$

it follows that Eq. (7) can also be rewritten as

$$\dot{\boldsymbol{\delta}} = \boldsymbol{\omega}_0 + \dot{\boldsymbol{v}}\theta \tag{9}$$

where $\boldsymbol{\omega}_0$ is the vector component of $\boldsymbol{\omega}$ evaluated along $\boldsymbol{v}$ and can be expressed as

$$\boldsymbol{\omega}_0 \triangleq [\boldsymbol{v}\boldsymbol{v}^T]\boldsymbol{\omega} \tag{10}$$

To complete the characterization of the time derivative of $\boldsymbol{\delta}$, let us proceed further by evaluating the time derivative of vector $\boldsymbol{v}$ appearing in Eq. (9). First differentiate the second of Eqs. (6) with respect to time, obtaining

$$(I - R)\dot{\boldsymbol{v}} = \dot{R}\boldsymbol{v} \tag{11}$$

Then, by substituting $\dot{R}$ with the right-hand side of Eq. (1), recalling the well-known properties of the (projected) vector cross-product operations, that is, $[\boldsymbol{\omega}\wedge]\boldsymbol{v} = -[\boldsymbol{v}\wedge]\boldsymbol{\omega}$, and premultiplying by R^T we get

$$\dot{\boldsymbol{v}} - R^T\dot{\boldsymbol{v}} = [\boldsymbol{v}\wedge]\boldsymbol{\omega} = [\boldsymbol{v}\wedge]\boldsymbol{\omega}_\perp \tag{12}$$

where $\boldsymbol{\omega}_\perp$ is defined as the vector component of $\boldsymbol{\omega}$ orthogonal to $\boldsymbol{v}$, given by

$$\boldsymbol{\omega}_\perp \triangleq (I - \boldsymbol{v}\boldsymbol{v}^T)\boldsymbol{\omega} = -[\boldsymbol{v}\wedge]^2\boldsymbol{\omega} \tag{13}$$

Refer to Fig. 1. Because $\boldsymbol{v}$ is a unit vector, then $\dot{\boldsymbol{v}}$ is orthogonal to $\boldsymbol{v}$; moreover, $R^T\dot{\boldsymbol{v}}$ (which has the same length as $\dot{\boldsymbol{v}}$) is also orthogonal to $\boldsymbol{v}$ (recall that R^T is the operator that rotates any vector of the angle θ clockwise about the vector $\boldsymbol{v}$). Hence, from Eq. (12), $[\boldsymbol{v}\wedge]\boldsymbol{\omega}_\perp$ is the third edge of the triangle lying on the plane orthogonal to $\boldsymbol{v}$ with two edges of equal length. Recall that $\boldsymbol{v}$ and $\boldsymbol{\omega}_\perp$ are orthogonal; then as regards the length of $\boldsymbol{\omega}_\perp$ we have

$$|\boldsymbol{\omega}_\perp| = 2|\dot{\boldsymbol{v}}|\sin(\theta/2) \tag{14}$$

and, moreover, $\dot{\boldsymbol{v}}$ obviously can be represented as

$$\dot{\boldsymbol{v}} = \text{vers}(\boldsymbol{\omega}_\perp)|\dot{\boldsymbol{v}}|\cos(\theta/2) + \text{vers}([\boldsymbol{v}\wedge]\boldsymbol{\omega}_\perp)|\dot{\boldsymbol{v}}|\sin(\theta/2) \tag{15}$$

where we have defined $\text{vers}(\boldsymbol{a}) = \boldsymbol{a}/|\boldsymbol{a}|$ as the unit vector of vector $\boldsymbol{a}$. Consequently, we have

$$\dot{\boldsymbol{v}} = \frac{1}{2}\left(I\frac{\cos(\theta/2)}{\sin(\theta/2)} + [\boldsymbol{v}\wedge]\right)\boldsymbol{\omega}_\perp \tag{16}$$

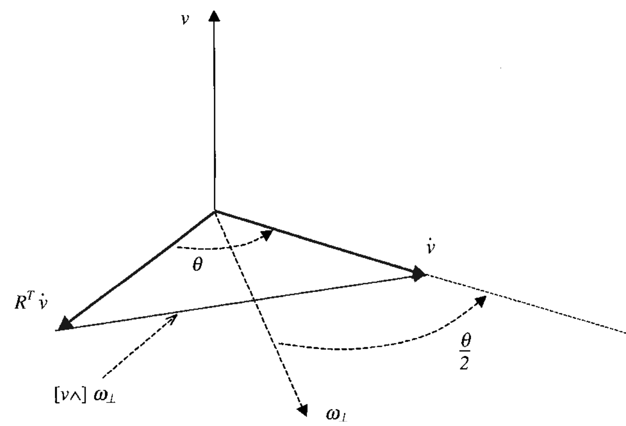

Fig. 1 Geometric relationships between $\boldsymbol{v}$, $\boldsymbol{\theta}$, $\boldsymbol{\omega}_\perp$, and $\dot{\boldsymbol{v}}$.

Thus, substituting Eq. (16) into Eq. (9), for the time derivative $\dot{\boldsymbol{\delta}}$ whenever $\theta \in (0, \pi)$, we finally get the equation

$$\dot{\boldsymbol{\delta}} = \boldsymbol{\omega}_0 + \frac{1}{2}\left(I\frac{\theta\cos(\theta/2)}{\sin(\theta/2)} + [\boldsymbol{\delta}\wedge]\right)\boldsymbol{\omega}_\perp \tag{17}$$

Note that the preceding analysis also might be done using quaternions.

Case 2: $\theta = \pi$

Equation (17) implicitly defines the relative motion of the two frames $\langle s \rangle$ and $\langle o \rangle$ when the angle $\theta \in (0, \pi)$. Let us now discuss the validity of Eq. (17) when $\theta = \pi$. In this case, problems can arise only in the correspondence of the behavior of θ. Recall that θ is measured so that it assumes only positive values; we can still use Eq. (17) only if θ tends toward the interior of its definition domain. This means that Eq. (17) is always true if $\boldsymbol{v}^T\boldsymbol{\omega} \leq 0$ for $\theta = \pi$.

On the other hand, if $\boldsymbol{v}^T\boldsymbol{\omega} > 0$ (which means that θ would tend to assume values greater that π), we should be in the presence of a switching condition, causing a discontinuous change of the direction of $\boldsymbol{\delta}$. Obviously, both $\dot{\boldsymbol{v}}$ and $\dot{\boldsymbol{\delta}}$ should have an impulsive behavior in correspondence with such a transition. Nevertheless, we might restart Eq. (17) with the new initial conditions

$$\theta\left(t_0^+\right) = \pi, \qquad \boldsymbol{v}\left(t_0^+\right) = -\boldsymbol{v}\left(t_0^-\right) \tag{18}$$

where t_0 is the time instant corresponding to such transition. This must be taken into account when selecting the proper value of $\boldsymbol{\omega}$ when defining the closed-loop control laws.

Case 3: $\theta = 0$

This case is in some sense more difficult to discuss because for $\theta = 0$ the vector $\boldsymbol{v}$ is not well defined. However, trajectories crossing $\theta = 0$ exist (as easy to understand from the definition of $\boldsymbol{\delta}$).

Let us consider a time evolution $\boldsymbol{\delta}(t)$ and a time interval $[t^* - \varepsilon, t^* + \varepsilon]$ (where ε is an arbitrary small number) such that

$$\boldsymbol{\delta}(t^*) = 0, \qquad \boldsymbol{\delta}(t) \in C^1 \; t \in [(t^* - \varepsilon, t^*) \cup (t^*, t^* + \varepsilon)] \tag{19}$$

Note that for t approaching t^* from both the left side and the right side, we have that θ approaches zero. Then because $\boldsymbol{\delta}(t) \in C^1$ in the corresponding subinterval,

$$\lim_{t \to t^*} \dot{\boldsymbol{\delta}} = \lim_{\theta \to 0} \dot{\boldsymbol{\delta}} = \boldsymbol{\omega}_0(t^*) + \boldsymbol{\omega}_\perp(t^*) \triangleq \boldsymbol{\omega}(t^*) \tag{20}$$

approaching t^* from either the left or the right side. However, for continuous and bounded angular velocities, θ is continuous as is $\dot{\boldsymbol{\delta}}$. Hence, for such angular velocities we can admit, as an analytic prolongation of Eq. (17) in $\theta = 0$, the equation

$$\dot{\boldsymbol{\delta}} = \boldsymbol{\omega} \tag{21}$$

that coincides with the same simple form assumed by Eq. (17) whenever $\boldsymbol{\omega}$ meets the condition $\boldsymbol{\omega}_0 = \boldsymbol{\omega}$, that is, $\boldsymbol{\omega}_1 = 0$. □

To conclude the modeling section, recall that, for any well-defined $\boldsymbol{\delta}$ and $\boldsymbol{\omega}$, $\dot{\boldsymbol{\delta}}$ has a component along $\boldsymbol{\delta}$ that is directly proportional to the component of $\boldsymbol{\omega}$ along $\boldsymbol{\delta}$. In fact,

$$\boldsymbol{\delta}^T\dot{\boldsymbol{\delta}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\boldsymbol{\delta}^T\boldsymbol{\delta} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\theta\boldsymbol{v}^T\theta\boldsymbol{v} = \theta\dot{\theta} = \theta\boldsymbol{v}^T\boldsymbol{\omega} = \boldsymbol{\delta}^T\boldsymbol{\omega} \tag{22}$$

Remark: Equation (22) suggests that if one would find a control signal $\boldsymbol{\omega}$ that drives the system to the desired attitude, that is, reaching the condition $\boldsymbol{\delta} = 0$, a possible approach is to look for $\boldsymbol{\omega}$ such that a negative time derivative of $\boldsymbol{\delta}^T\boldsymbol{\delta}$ is obtained, thereby assuring a monotonic convergence of $\boldsymbol{\delta}$ to zero.

Closed-Loop Approach to Kinematic Attitude Control

Let the spacecraft fixed frame $\langle s\rangle$ be initially located at any attitude $R(0)$ with respect to the reference frame $\langle o\rangle$. Then, by considering $\boldsymbol{\omega}$ as the input control vector, we look for closed-loop control laws capable of asymptotically driving R toward the identity matrix.

Unconstrained Case

Consider the simple case in which $\boldsymbol{\omega}$ can be freely chosen in a three-dimensional space. In such a situation, let us consider the following positive definite quadratic form as a candidate Lyapunov function:

$$V = \tfrac{1}{2}\boldsymbol{\delta}^T\boldsymbol{\delta} = \tfrac{1}{2}\theta^2 \tag{23}$$

V measures the squared norm of the orientation error between $\langle s\rangle$ and $\langle o\rangle$, which is properly represented by the equivalent angle-axis vector $\boldsymbol{\delta}$. By differentiating with respect to time within the domain $\theta \in (0, \pi)$, we have

$$\dot{V} = \boldsymbol{\delta}^T\dot{\boldsymbol{\delta}} = \boldsymbol{\delta}^T\boldsymbol{\omega} \tag{24}$$

Then, in all of the cases where $\boldsymbol{\omega}$ can be arbitrarily assigned, it is easy to solve the attitude control problem by simply choosing $\boldsymbol{\omega}$ of the form

$$\boldsymbol{\omega} = -\gamma\boldsymbol{\delta}, \qquad \gamma > 0 \tag{25}$$

with γ not necessarily constant, and where the orientation error $\boldsymbol{\delta}$ can be directly evaluated, on line, via the application of the following well-known formulas (Versors's lemma)

$$R - R^T = 2[\boldsymbol{v}\wedge]\sin\theta \tag{26}$$

$$\mathrm{tr}(R) = 1 + 2\cos\theta \tag{27}$$

Control law (25) gives rise to a $\dot{V}$ that is negative definite within the open domain $\theta \in (0, \pi)$, that is,

$$\dot{V} = -\gamma\boldsymbol{\delta}^T\boldsymbol{\delta} = -\gamma\theta^2 \tag{28}$$

Then, the norm θ exponentially decreases to zero, thus ensuring the convergence of $\boldsymbol{\delta}$ to zero and of R to I. Moreover, from the resulting closed-loop equations

$$\dot{\boldsymbol{\delta}} = -\gamma\boldsymbol{\delta} \tag{29}$$

we can see how that the asymptotic convergence of $\boldsymbol{\delta}$ toward zero always proceeds monotonically (thus, without any finite-time zero crossing or zero reaching) along the same direction exhibited by $\boldsymbol{\delta}(0)$ whenever it is located outside of the origin.

Note how the application of control law (25), and consequently the validity of expression (28), can actually be extended to the whole closed domain $\theta \in [0, \pi]$ because, with control law (25), a locally decreasing behavior for θ turns out to be ensured even in correspondence of $\theta = \pi$ and, from Eq. (29), $\boldsymbol{\delta}(t) \in C^\infty$.

Constrained Control

In this section it will be apparent how the geometric properties of the equivalent angle axis can be used to devise in an intuitive way the control laws to be applied in the presence of a nonholonomic constraint.

Let us consider the case when $\boldsymbol{\omega}$ is constrained to lie on one of the coordinate planes of $\langle s\rangle$ (for instance, the x–y plane). Under this condition, obviously control law (25) can be used in its present form (also inducing the same behavior and asymptotic properties) only within the very particular case of $\boldsymbol{\delta}(0)$ exactly lying on the constraint plane x–y, at the initial time.

If this were not the case, however, we might try to approximate Eq. (25) by simply projecting $\boldsymbol{\delta}$ on the x–y plane, thus tentatively proposing a control law of the form

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 \triangleq -\gamma(I - \boldsymbol{k}\boldsymbol{k}^T)\boldsymbol{\delta} = \gamma[\boldsymbol{k}\wedge]^2\boldsymbol{\delta}, \qquad \gamma > 0 \tag{30}$$

with $\boldsymbol{k} = [0, 0, 1]^T$ the unitary vector of the z axis of $\langle s\rangle$.

Now define α as the angle formed by vector $\boldsymbol{\delta}$ with its orthogonal projection on the x–y plane, measured starting from the x–y plane. Therefore, $\alpha \in [0, \pi/2]$ is well defined only for $\delta \neq 0$ and is arbitrary but not meaningful for $\theta = 0$. Using Eq. (30) in Eq. (24) and first assuming $\theta \in (0, \pi)$, we have

$$\dot{V} = \boldsymbol{\delta}^T\boldsymbol{\omega}_1 = \gamma\boldsymbol{\delta}^T[\boldsymbol{k}\wedge]^2, \qquad \boldsymbol{\delta} = -\gamma(\cos^2\alpha)\theta^2 \leq 0 \tag{31}$$

From Eq. (8)

$$\dot{\theta} = -\gamma(\cos^2\alpha)\theta \tag{32}$$

Remark: Expression (32) indicates that θ is always nonincreasing. Hence, both Eqs. (31) and (32) can be extended in the whole range $\theta \in [0, \pi]$. Note that it is possible to deduce an exponential decreasing to zero for θ if

$$-\gamma(\cos^2\alpha) \leq -\xi < 0 \tag{33}$$

□

The presence of the bounded quantity $\cos^2\alpha$ appearing in Eq. (31) makes $\dot{V}$ only a generally time-varying [because it is dependent on the time evolution of angle $\alpha(t)$ acting as an input], negative, semidefinite form within the domain $\theta \in [0, \pi]$ (instead of being unconditionally negative definite as in the unconstrained case). This means that, generally, even if a nonincreasing evolution for the norm θ of $\boldsymbol{\delta}$ within $[0, \pi]$ is guaranteed the asymptotic attainment of a null value by part of θ itself might instead not occur [obviously apart from the trivial case when $\boldsymbol{\delta}(0) = 0$]. In fact, by examination of expression (31), it is an easy matter to verify that any vector $\bar{\boldsymbol{\delta}}$ located on the z axis (hence defining an angle α equal to $\pi/2$) and with norm $\bar{\theta} \in [0, \pi]$ might actually result in a possible convergence point for the whole closed-loop system. This means that after some time the controlled system may find an equilibrium point that is not the desired one, that is, characterized by a wrong attitude.

To clearly understand the behavior of the system, consider that, from the definition of α, the quantity $\dot{\alpha}$ is given by the component of $\dot{\boldsymbol{v}}$ directed along a unit vector $\boldsymbol{b}$ orthogonal to $\boldsymbol{v}$, lying on the plane containing $\boldsymbol{v}$ and $\boldsymbol{k}$, and directed out of the x–y plane. Obviously, $\boldsymbol{b}$ is uniquely defined only if $\alpha \neq 0$ and $\pi/2$. More formally, if $\alpha \neq 0$ and $\pi/2$, then

$$\sin\alpha = \boldsymbol{k}^T\boldsymbol{v}, \qquad \text{if} \qquad \boldsymbol{k}^T\boldsymbol{v} > 0 \tag{34}$$

$$\sin\alpha = -\boldsymbol{k}^T\boldsymbol{v}, \qquad \text{if} \qquad \boldsymbol{k}^T\boldsymbol{v} < 0 \tag{35}$$

Differentiating with respect to time and using Eq. (16), we get

$$\dot{\alpha} = \frac{1}{2\cos\alpha}\boldsymbol{k}^T\left(I\frac{\cos(\theta/2)}{\sin(\theta/2)} + [\boldsymbol{v}\wedge]\right)\boldsymbol{\omega}_{1\perp}, \qquad \text{if} \quad \boldsymbol{k}^T\boldsymbol{v} > 0 \tag{36}$$

$$\dot{\alpha} = -\frac{1}{2\cos\alpha}\boldsymbol{k}^T\left(I\frac{\cos(\theta/2)}{\sin(\theta/2)} + [\boldsymbol{v}\wedge]\right)\boldsymbol{\omega}_{1\perp}, \qquad \text{if} \quad \boldsymbol{k}^T\boldsymbol{v} < 0 \tag{37}$$

Note that for $\alpha = 0$ just one of the two preceding equations can be used, that is, the one yielding $\dot{\alpha} > 0$ (this is due to the assumed way of measuring α).

If we now consider the use of control law (30) it is easy to see, with the help of Fig. 2, that $\dot{\alpha}$ is given by the component of $\dot{\boldsymbol{v}}$ on the

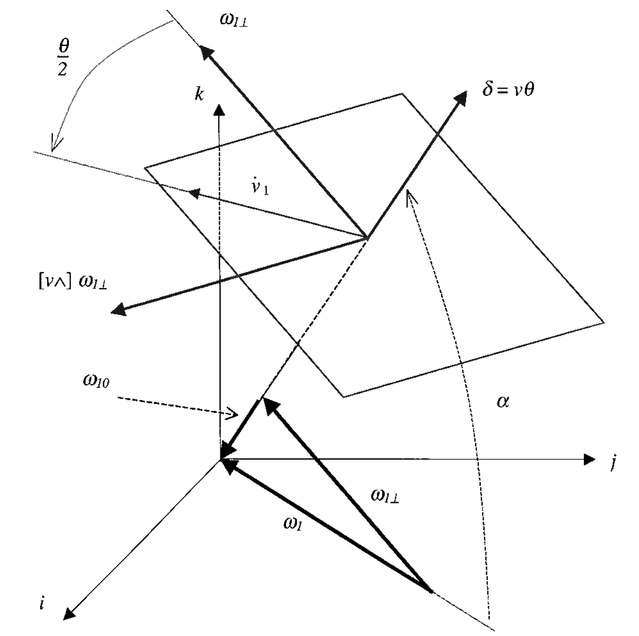

Fig. 2 Geometric representation of the control signal ω_1 and related quantities.

plane where α is defined, that is, $\dot{\alpha} = |\dot{\boldsymbol{v}}| \cos(\theta/2)$. Then using formula (14), both Eqs. (36) and (37) collapse into a single differential equation, that is,

$$\dot{\alpha} = \frac{1}{2}|\boldsymbol{\omega}_{1\perp}|\frac{\cos(\theta/2)}{\sin(\theta/2)} = \frac{\gamma}{2}(\sin\alpha\cos\alpha)\frac{\theta}{\sin(\theta/2)}\cos\left(\frac{\theta}{2}\right) \quad (38)$$

where $\boldsymbol{\omega}_{1\perp}$ is the vector component of $\boldsymbol{\omega}_1$ orthogonal to $\boldsymbol{v}$.

As apparent in the defined domains of α and θ, angle $\dot{\alpha}$ is always greater than zero, generally inducing a monotonic increase of α toward $\pi/2$, except for the single case corresponding to $\alpha(0) = 0$.

A descriptive analysis of what generally occurs within the adoption of a control law of type (30) can be performed with the help of Fig. 2. In fact, in the case of $\boldsymbol{\delta} \neq 0$, the vector component $\boldsymbol{\omega}_{1_o}$ of $\boldsymbol{\omega}_1$ that is directed as $-\boldsymbol{\delta}$ is only responsible for the nonincreasing behavior of the norm of $\boldsymbol{\delta}$. Furthermore, $\boldsymbol{\omega}_{1_o}$ does not affect $\dot{\boldsymbol{v}}$. On the other hand, the orthogonal component $\boldsymbol{\omega}_{1\perp}$, provided it is nonzero at the given instant (i.e., both $\alpha \neq 0$ and $\pi/2$), only induces a nonzero time derivative $\dot{\boldsymbol{v}}_1$ to $\boldsymbol{v}$, as established by Eq. (16) with specifications (30). Then, due to the presence of such a nonzero time derivative $\dot{\boldsymbol{v}}_1$, we have that its corresponding part aligned with $\boldsymbol{\omega}_{1\perp}$ naturally induces a motion component on $\boldsymbol{v}$, and then on $\boldsymbol{\delta}$, directed toward the z axis. This in turn gives rise to a motion for α directed toward $\pi/2$. Such a situation will lead to the eventual (as we shall discuss later) monotonic and asymptotic convergence of angle α toward $\pi/2$. This will also imply that δ will be asymptotically aligned with $\boldsymbol{k}$ if initially located strictly over the x–y plane, or asymptotically aligned with $-\boldsymbol{k}$ in the opposite case.

The remaining part of $\dot{\boldsymbol{v}}_1$, that is, the one orthogonal to the former part and directed as $[\boldsymbol{v}\wedge]\boldsymbol{\omega}_{1\perp}$, instead gives rise to a motion component, for both $\boldsymbol{v}$ and $\boldsymbol{\delta}$, that is exclusively of precession type around the z axis (of clockwise type in the upper half-space and counterclockwise in the lower one).

Note that control law (30) cannot instead introduce any modification on $\boldsymbol{\delta}$ whenever $\boldsymbol{\delta}(0) \neq 0$ is exactly aligned with the z axis (thus letting $\alpha = \pi/2$ for any $t \geq 0$). On the contrary, control law (30) behaves as for the unconstrained case whenever $\boldsymbol{\delta}(0) \neq 0$ and $\alpha(0) = 0$.

Qualitatively, the motions $\boldsymbol{\delta}(t)$ that can be expected are of the kind shown in Fig. 3. A formal proof can now be sketched by considering the scalar function

$$H = \tfrac{1}{2}(\pi/2 - \alpha)^2 \quad (39)$$

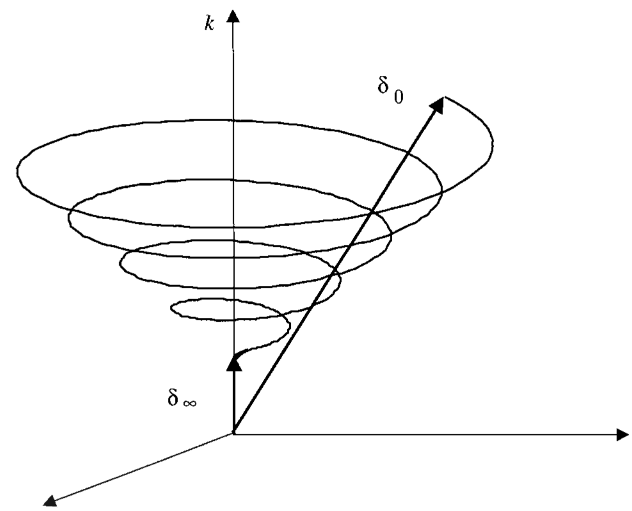

Fig. 3 Typical precession motion due to the control signal ω_1.

Taking the time derivative along Eq. (38), we have

$$\dot{H} = -(\gamma/2)(\sin\alpha\cos\alpha)[\theta/\sin(\theta/2)]\cos(\theta/2) \leq 0$$
$$\text{for} \quad \alpha \in (0, \pi/2), \qquad \theta \in (0, \pi) \quad (40)$$

that vanishes in correspondence of $\{\theta = \pi, \forall\alpha\} \cup \{\alpha = 0; \alpha = \pi/2; \forall\theta\}$. Note that, because θ is nonincreasing [from Eq. (32)] the first equilibrium point of $\dot{H}$ could be attained only if $\theta = \pi\,\forall t$, and, hence, α necessarily would be equal to $\pi/2$. For any other θ, the condition $\alpha = 0$ results to be an unstable equilibrium point, whereas $\alpha = \pi/2$ is a stable convergence point. As a consequence of the preceding analysis, the need arises to try to modify the control law (30) so that $\boldsymbol{\delta}$ has nonincreasing norm θ and α converges asymptotically and monotonically toward zero. The convergence of α in turn will ensure the convergence to zero of $\boldsymbol{\delta}$.

To attempt a possible solution to the outlined need, let us again start by reconsidering the case when $\boldsymbol{\delta}(0) \neq 0$, while also preliminarily assuming $\alpha(0) \in (0, \pi/2)$. Let us consider the following modified form of the control law (30):

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \quad (41)$$

with $\boldsymbol{\omega}_1$ as in Eq. (30) and $\boldsymbol{\omega}_2$ still to be specified, but of the form

$$\boldsymbol{\omega}_2 = \boldsymbol{u}h(\cdot), \qquad h(\cdot) \geq 0 \quad (42)$$

where

$$\boldsymbol{u} = \boldsymbol{v}\wedge\boldsymbol{k}/\cos\alpha \qquad \text{if} \qquad \boldsymbol{k}^T\boldsymbol{v} > 0 \quad (43)$$

$$\boldsymbol{u} = -\boldsymbol{v}\wedge\boldsymbol{k}/\cos\alpha \qquad \text{if} \qquad \boldsymbol{k}^T\boldsymbol{v} < 0 \quad (44)$$

Therefore, $\boldsymbol{u}$ is the unitary vector directed as the opposite of the vector product between $\boldsymbol{\delta}$ and its projection $-[\boldsymbol{k}\wedge]^2\boldsymbol{\delta}$ on the x–y plane; obviously $\boldsymbol{u}$ is well defined if $\boldsymbol{\delta} \neq 0$ and $\alpha \in (0, \pi/2)$.

The rationale underlying the proposed modification, for the assumed case, can be understood with the help of Fig. 4. First note that whenever $\boldsymbol{\omega}_{\perp 1}$ is nonzero it lies on the unit vector $\boldsymbol{b}$ defined before. In Fig. 4, it can be seen that $\boldsymbol{\omega}_2$, which is orthogonal to $\boldsymbol{v}$, cannot influence the rate of variation of the norm of $\boldsymbol{\delta}$. In this way the expression for $\dot{V}$ given by Eq. (31) still remains valid also using the control law (41). However, $\boldsymbol{\omega}_2$ induces an additional contribution $\dot{\boldsymbol{v}}_2$ to that, $\dot{\boldsymbol{v}}_1$ due to $\boldsymbol{\omega}_{1\perp}$. Because the components along $\boldsymbol{\omega}_{1\perp}$ of both $\dot{\boldsymbol{v}}_2$ and $\dot{\boldsymbol{v}}_1$ just result with opposite directions, this should leave the possibility of motion of the angle α directed toward zero, instead of $\pi/2$.

Furthermore, we require that $h(\cdot) = 0$ for $\alpha = 0$: In fact, if δ is exactly located on the x–y plane, control law (30) would make $\boldsymbol{\delta}$ converge to zero by itself, and then it would not make sense to perturb such a good situation. We can now better formalize this problem of making angle α converge to zero (via the use of the suggested modified control law), for any well-defined angle $\alpha \in (0, \pi/2)$, that is, in correspondence of any $\boldsymbol{\delta} \neq 0$ neither lying on the z axis nor on the x–y plane.

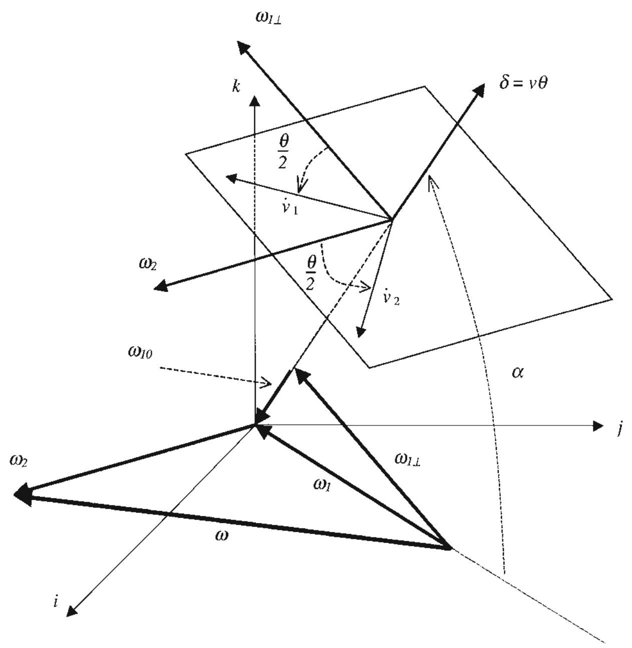

Fig. 4 Geometric representation of the control signals ω_1 and ω_2 and related quantities.

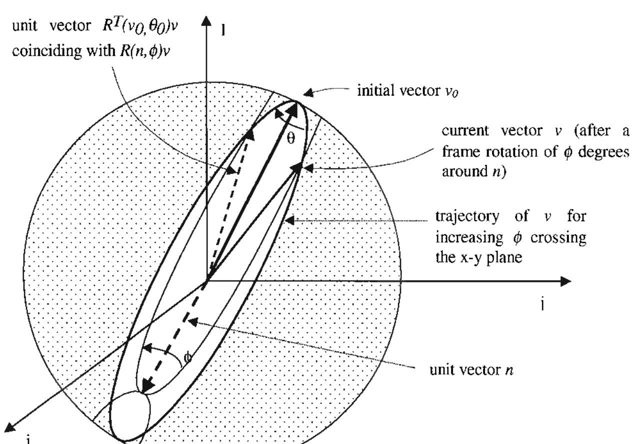

Fig. 5 Motion of *v* corresponding to a finite rotation of ⟨*s*⟩ about *n*.

First note that $\omega_\perp = \omega_{1\perp} + \omega_2$. By the taking into account of the definition of $\boldsymbol{u}$ and Eqs. (36) [(or Eq. (37)] and (38) the following equation holds true, independently from the values assigned to the variable h appearing in Eq. (42):

$$\dot{\alpha} = \frac{1}{2}\left(|\omega_{1\perp}|\frac{\cos(\theta/2)}{\sin(\theta/2)} - h\right) = \frac{1}{2}\left(\gamma(\sin\alpha\cos\alpha)\frac{\theta}{\sin(\theta/2)}\cos\left(\frac{\theta}{2}\right) - h\right) \tag{45}$$

By now introducing the positive definite quadratic function

$$U = \tfrac{1}{2}\alpha^2 \tag{46}$$

and differentiating it with respect to time along Eq. (45), we obtain

$$\dot{U} = \frac{1}{2}\alpha\left(|\omega_{1\perp}|\frac{\cos(\theta/2)}{\sin(\theta/2)} - h\right) = \frac{1}{2}\alpha\left(\gamma(\sin\alpha\cos\alpha)\frac{\theta}{\sin(\theta/2)}\cos\left(\frac{\theta}{2}\right) - h\right) \tag{47}$$

which shows that, if we choose h of the form

$$h = \mu\alpha + \gamma(\sin\alpha\cos\alpha)[\theta/\sin(\theta/2)]\cos(\theta/2), \qquad \mu > 0 \tag{48}$$

we immediately obtain

$$\dot{U} = -\tfrac{1}{2}\mu\alpha^2 < 0 \tag{49}$$

and, consequently, by substituting Eq. (48) in Eq. (45), we obtain

$$\dot{\alpha} = -(\mu/2)\alpha \tag{50}$$

which implies an exponentially decreasing behavior to $\alpha(t)$, entirely evolving within the left open domain $\alpha \in (0; \alpha(0) < \pi/2]$ for any $t \geq 0$, together with an asymptotic convergence toward zero for any choice of μ (also state dependent) such that $\mu \geq \zeta > 0$. As a consequence, the monotonic and asymptotic convergence of θ toward zero, as well as that of $\boldsymbol{\delta}$, is in turn guaranteed. In fact, by defining the complete Lyapunov candidate function

$$W = \tfrac{1}{2}\boldsymbol{\delta}^T\boldsymbol{\delta} + \tfrac{1}{2}\alpha^2 \tag{51}$$

the time derivative of W corresponding to the use of Eq. (41) results in

$$\dot{W} = -\gamma(\cos^2\alpha)\theta^2 - \tfrac{1}{2}\mu\alpha^2 < 0 \tag{52}$$

which is negative definite and then assures the asymptotic convergence to zero of both α and θ.

Furthermore, h, as given by Eq. (48), actually results in a bounded function of variables θ and α that is also continuous and differentiable within the considered domain $\theta \in (0, \pi]$, $\alpha \in (0; \pi/2)$.

In conclusion, the complete control law results,

$$\omega = \omega_1 + \omega_2 = \omega_1 + \boldsymbol{u}h = \gamma[\boldsymbol{k}\wedge]^2\boldsymbol{\delta} + \boldsymbol{u}\{\mu\alpha + \gamma(\sin\alpha\cos\alpha)[\theta/\sin(\theta/2)]\cos(\theta/2)\} \tag{53}$$

where for the unitary vector $\boldsymbol{u}$ we have

$$\boldsymbol{\delta} \neq 0 \text{ and } \alpha \in (0, \pi/2) \rightarrow \begin{cases} \boldsymbol{u} = \boldsymbol{v}\wedge\boldsymbol{k} & \text{if } \boldsymbol{k}^T\boldsymbol{v} > 0 \\ \boldsymbol{u} = -\boldsymbol{v}\wedge\boldsymbol{k} & \text{if } \boldsymbol{k}^T\boldsymbol{v} < 0 \end{cases} \tag{54}$$

Now consider the case when $\boldsymbol{\delta}(0) \neq 0$ and $\alpha(0) = 0$, as already mentioned and expected, no modifications, that is, $\omega_2 = 0$, were actually needed for control law (30) because angle α would remain zero for all future instants [thus allowing Eq. (30) itself to behave as in the unconstrained ω case]. In fact, we note that the control law given by Eq. (53) becomes the one given by Eq. (30) whenever $\alpha = 0$. Therefore, we could naturally extend the validity of Eq. (53) to the whole domain $\theta \in (0, \pi]$, $\alpha \in [0, \pi/2)$, by simply letting $\boldsymbol{u}$ be any arbitrary (because it is not influent) unitary vector orthogonal to $\boldsymbol{k}$, whenever $\alpha = 0$.

The same kind of extension also could be directly applied to the case $\boldsymbol{\delta}(0) = 0$, that is, when no corrections were needed to control law (30), in this case generating a persistently null constant action, provided that we accept defining $\alpha \triangleq 0$ in correspondence with $\theta = 0$.

Finally, by considering the last situation, when $\boldsymbol{\delta}(0) \neq 0$ and $\alpha(0) = \pi/2$, we must recall that ω_1, generated by Eq. (30), would be actually zero at $t = 0$ (thus not producing any local decreasing behavior on the norm θ of $\boldsymbol{\delta}$ at $t = 0$). Furthermore, the simultaneous action of the nonnull additional term ω_2 should be necessary to perturb $\boldsymbol{\delta}$ and to move it away from its original direction aligned with the z axis. If $\alpha(0) = \pi/2$, vector $\boldsymbol{u}$ is not well defined. Therefore, consider $\boldsymbol{u}$ as any unit vector orthogonal to $\boldsymbol{k}$ (and, hence, to $\boldsymbol{v}$ in this case). Because h is not zero, the system will evolve in such a way that after any small time interval the angle α will be less than $\pi/2$. Therefore, after that interval, both θ and α will be asymptotically convergent toward zero.

Hence, the control law given in Eq. (53) can be considered within the complete domains $\theta \in [0, \pi]$ and $\alpha \in [0, \pi/2]$, provided we define the following choice for $\boldsymbol{u}$:

$$\boldsymbol{\delta} \neq 0 \text{ and } \alpha \in (0, \pi/2) \rightarrow \begin{cases} \boldsymbol{u} = \boldsymbol{v} \wedge \boldsymbol{k} & \text{if } \quad \boldsymbol{k}^T \boldsymbol{v} > 0 \\ \boldsymbol{u} = -\boldsymbol{v} \wedge \boldsymbol{k} & \text{if } \quad \boldsymbol{k}^T \boldsymbol{v} < 0 \end{cases} \tag{55}$$

$$\alpha = 0 \text{ or } \alpha = \pi/2 \rightarrow \boldsymbol{u} \text{ is any unitary vector orthogonal to } \boldsymbol{k} \tag{56}$$

□

Analysis of the Closed-Loop Motion in the Constrained Case

Consider now the problem of analyzing what we earlier called precession type motion about the z axis that is generally imposed to all vectors by Eq. (53), whenever operating within the conditions $\boldsymbol{\delta} \neq 0$ and $\alpha \in [0, \pi/2)$, and basically due to the components of $\dot{\boldsymbol{v}}$ aligned with $[\boldsymbol{v} \wedge]\boldsymbol{\omega}_\perp$, (see also Fig. 5).

Let Ω be the scalar angular velocity characterizing such a motion (still clockwise in the upper half-space, counterclockwise in the lower, and obviously null for motions starting, and then entirely evolving, on the x–y plane). Ω is the component of the total derivative $\dot{\boldsymbol{v}}$ aligned with $\boldsymbol{u}$ and divided by the orthogonal component of $\boldsymbol{v}$ with respect to the unit vector $\boldsymbol{k}$ that is

$$\Omega = \dot{\boldsymbol{v}}^T \boldsymbol{u} / \cos\alpha \tag{57}$$

The use of Eq. (16) and of some vector algebra leads to

$$\Omega = \frac{1}{2\cos\alpha}\left(|\boldsymbol{\omega}_{1\perp}| + \frac{\cos(\theta/2)}{\sin(\theta/2)} |\boldsymbol{\omega}_1| \right) \tag{58}$$

that, using Eq. (53), immediately becomes

$$\Omega = \frac{\gamma}{2} \frac{\theta \sin\alpha}{2\sin^2(\theta/2)} + \frac{\mu}{2} \frac{\alpha}{\cos\alpha} \frac{\cos(\theta/2)}{\sin(\theta/2)} \tag{59}$$

which evidences that, for $\boldsymbol{\delta}$ converging to zero from outside the x–y plane (otherwise Ω would be identically zero), Ω could tend to infinity, tend to a constant, or could be monotonically and asymptotically convergent to zero, depending of the exponential convergence rate assigned to α with respect to that of θ. Because the convergence rate of α only depends on the gain factor μ, while that of θ is mainly influenced by γ, it is clear, from Eq. (59) that by choosing μ sufficiently larger than γ, we can always guarantee that Ω will be also monotonically and asymptotically convergent to zero.

As it can be easily realized, the case when Ω also converges to zero actually corresponds to a terminal motion for vector $\boldsymbol{\delta}$ where its tip, monotonically and asymptotically, reaches the origin by landing on the x–y plane, eventually proceeding toward zero with an asymptotically constant final direction.

From a practical point view, a choice for gains γ and μ as already indicated represents a great advantage because it allows the adoption of control actions $\boldsymbol{\omega}(t)$ characterized by monotonically reducing spinning rates $\Omega(t)$ for increasing time, that is, with a reducing angular acceleration vector for increasing time.

Unfortunately such a nice property is generally counterbalanced by a nonnegligible drawback that might occur at very early instants of the application of control law (53). In fact from Eq. (59), it appears that Ω tends to be unbounded whenever 1) α is very close to the $\pi/2$ value and/or 2) the ratios $\sin\alpha/\sin(\theta/2)$ and $\alpha/\sin(\theta/2)$ are high (possibly even with small value for α).

Consequently, it is clear that, when starting with $\boldsymbol{\delta}(0) \neq 0$ and $\alpha(0) \in (0, \pi/2)$ and falling within at least one of the outlined cases (1 and/or 2), extremely high-rate spinning angular velocity vectors $\boldsymbol{\omega}(t)$ (also inducing extremely high angular accelerations) would actually be required at the very beginning of the application of control law (53).

Such a negative property would prevent the application of Eq. (53) in a real environment where vehicle dynamics have to be taken into account and infinite reference signals obviously cannot be admitted. For instance, consider the hierarchical control architecture proposed in Ref. 9 consisting of an outer loop where a robotic system is considered at a kinematic level and of an inner loop acting on the dynamics and devoted to follow the reference velocities provided by the outer loop. To assure the effectiveness of the inner loop, one has to require the boundedness of the reference signals and of their derivatives.

For these reasons, the following analysis will be devoted to design operating modes avoiding the generation of infinite angular accelerations. Let us first upper bound the right-hand side of Eq. (59) considering $\mu = 2\sigma\gamma$, that is,

$$\Omega \leq K\gamma(1/\cos\alpha)(\alpha/\theta) \tag{60}$$

where K is a suitable constant value. We can immediately see that, to guarantee a spinning rate Ω always lower than an assigned maximum one Ω_m, we should have the condition

$$\gamma \leq \Omega_m(\theta\cos\alpha/K\alpha) \tag{61}$$

Such a condition can be trivially fulfilled by choosing

$$\gamma = \min\left\{\Omega_m\theta\cos\alpha/K\alpha,\ \gamma^*\right\} \tag{62}$$

where $\gamma^* > 0$ represents the maximum allowable feedback gain. This expression leads to the following closed-loop equations for α and θ:

$$\dot{\theta} = -\min\left\{\Omega_m\theta\cos\alpha/K\alpha,\ \gamma^*\right\}(\cos^2\alpha)\theta \tag{63}$$

$$\dot{\alpha} = -\sigma\min\left\{\Omega_m\theta\cos\alpha/K\alpha,\ \gamma^*\right\}\alpha \tag{64}$$

Note that the region where the gain parameters become state independent is described by

$$\theta\cos\alpha/\alpha \geq \gamma^*(K/\Omega_m) \tag{65}$$

For the behavior of θ with respect to α, we can study the time evolution of the positive scalar function

$$F = (\alpha/\theta)^2 \tag{66}$$

Taking the time derivative along Eqs. (63) and (64), we get

$$\dot{F} = -(\alpha/\theta)^2\min\left\{\Omega_m\theta\cos\alpha/K\alpha,\ \gamma^*\right\}(\sigma - \cos^2\alpha) \tag{67}$$

which is well defined and negative for $\theta \in (0, \pi]$ and $\alpha \in (0, \pi/2)$ if $\sigma > 1$. Such a study shows that, for θ and α initially within the given domains, the ratio α/θ has a decreasing behavior allowing us to conclude that

$$\lim_{t\to\infty}(\alpha/\theta) = 0 \tag{68}$$

which also implies, together with the monotonic decreasing behavior of α and θ, that after a transient time interval such that α and θ fulfill Eq. (65), the gain parameter will remain stationary.

Moreover, the time evolution of the gain parameters can be put into evidence by noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\theta\cos\alpha}{\alpha} = \frac{\gamma\theta\alpha\cos\alpha(\sigma - \cos^2\alpha) + \sigma\gamma\theta\alpha^2\sin\alpha}{\alpha^2} \tag{69}$$

which is well defined and positive for $\theta \in (0, \pi]$ and $\alpha \in (0, \pi/2)$ if $\sigma > 1$. This means that the gain parameters are monotonically nondecreasing with time.

For the boundary conditions it is easy to see, from the closed-loop equations, that problems can arise only in correspondence of $\alpha = \pi/2$, which defines an undesired equilibrium point of the descriptive variables. From a practical point of view, the gain parameters expressed by formula (62) generate for $\alpha = \pi/2$ an undesired equilibrium point and induce for α close to $\pi/2$, or small ratios $\theta/2$, an extremely slow motion of the system.

Preliminary Maneuvering

To overcome the mentioned drawback, a preliminary maneuvering action is proposed. The goal of such operation is to transfer the variables α and θ to locations suitable for the subsequent application of Eq. (53) with reasonable convergence velocity, namely, values such that condition expressed by Eq. (65) is fulfilled.

As apparent from Eq. (59), and because the gains are assured to be monotonically nondecreasing by Eq. (69), the slowest convergence velocity of the system toward the goal is a function of the initial conditions through

$$\gamma(0) = \min\left\{\frac{\Omega_m \theta(0)\cos\alpha(0)}{K\alpha(0)}, \gamma^*\right\} \tag{70}$$

It may happen that the immediate application of Eq. (53) could give rise to excessively slow motion. If we want to apply Eq. (53) being sure that

$$\gamma(t) \geq \rho\gamma^*, \qquad \rho \in (0, 1) \tag{71}$$

we must decompose the control problem into two phases, that is, phase 1, if necessary drive smoothly α and θ in such a way that Eq. (71) holds in some finite time t^*, and phase 2, from t^* on apply Eq. (53).

Phase 1 is what we call preliminary maneuvering. Recall that the need of this phase depends on the initial values of α and θ. To explain how to implement the preliminary maneuver consider the following.

Lemma: Let $\alpha_0 \triangleq \alpha(0) \neq 0$ and $\theta_0 \triangleq \theta(0) \neq 0$ be the values attained by the angles relevant to orientation vector $\boldsymbol{\delta}_0 \triangleq \boldsymbol{\delta}(0) \neq 0$ at the initial time. Consider any unitary vector $\boldsymbol{n}$ satisfying the condition $\boldsymbol{n}^T \boldsymbol{v}_0 \in (-1, 1)$, that is, not perfectly aligned with $\boldsymbol{v}_0$; also refer to an angular rotation ϕ along such vector $\boldsymbol{n}$, superimposed to the spacecraft frame $\langle s\rangle$ starting with initial orientation $\boldsymbol{\delta}_0$, by an angular velocity $\boldsymbol{\omega}$ always directed along $\boldsymbol{n}$. Then, the following holds true 1) $\exists\, \phi^0 \in (0, 2\pi]$ such that $\alpha(\phi^0) = 0$ and 2) $\theta(\phi) \neq 0$.

Proof: Let $R(v_0\theta_0)$ be the rotation matrix at the initial time instant. After a rotation of amplitude ϕ (without loss of generality assume it counterclockwise), along a unit vector $\boldsymbol{n}$ the new rotation matrix is given by

$$R(\theta\boldsymbol{v}) = R(\boldsymbol{v}_0\theta_0)R(\boldsymbol{n}, \phi) \tag{72}$$

The new equivalent angle axis $\boldsymbol{v}$ must satisfy Eq. (6), that is,

$$[I - R(\boldsymbol{v}_0\theta_0)R(\boldsymbol{n}, \phi)]\boldsymbol{v} = 0 \tag{73}$$

By premultiplying Eq. (73) by $R^T(\boldsymbol{v}_0\theta_0)$, we obtain

$$\left[R^T(\boldsymbol{v}_0\theta_0) - R(\boldsymbol{n}, \phi)\right]\boldsymbol{v} = 0 \tag{74}$$

which can be rewritten as

$$R^T(\boldsymbol{v}_0\theta_0)\boldsymbol{v} = R(\boldsymbol{n}, \phi)\boldsymbol{v} \tag{75}$$

Equation (75) states the following: The new unit vector $\boldsymbol{v}$ must be such that its clockwise rotation about $\boldsymbol{v}_0$ coincides with its counterclockwise rotation about $\boldsymbol{n}$. By definition of rotation about a given vector, the vector

$$\boldsymbol{\Delta} = \boldsymbol{v} - R(\boldsymbol{n}, \phi)\boldsymbol{v} = \boldsymbol{v} - R^T(\boldsymbol{v}_0\theta_0)\boldsymbol{v} \tag{76}$$

must be orthogonal both to $\boldsymbol{n}$ and $\boldsymbol{v}_0$. Hence, we can better understand the behavior of $\boldsymbol{v}$ by considering the plane defined by $\boldsymbol{v}_0$ and $\boldsymbol{n}$ (refer to Fig. 5). Any couple of unit vector $\boldsymbol{v}$ and $R^T(\boldsymbol{v}_0\theta_0)\boldsymbol{v}$ (that must rotate clockwise of θ_0 around $\boldsymbol{v}_0$) must have its tip belonging to the two meridians of the unity sphere with pole $\boldsymbol{v}_0$ defined by a solid angle θ_0. In the same way, any couple of unit vector $\boldsymbol{v}$ and $R(n\phi)\boldsymbol{v}$ (that must rotate counterclockwise of ϕ around $\boldsymbol{n}$) must have its edges belonging to the two meridians with pole $\boldsymbol{n}$ defined by a solid angle ϕ. Hence, for different values of ϕ, unit vector $\boldsymbol{v}$ will proceed along one of the meridians of $\boldsymbol{v}_0$ until, for $\phi = 2\pi$, it coincides again with $\boldsymbol{v}_0$. Now, because any meridian of $\boldsymbol{v}_0$ crosses the x–y plane (for instance, see Fig. 5) it proves for some ϕ^0 unit vector $\boldsymbol{v}$ will lie on such a plane, thereby showing part 1 of the lemma. □

For part 2 of the lemma, by definition, $R(\boldsymbol{v}_0\theta_0)$ is the only rotation (about a fixed axis) that can reorient the frame $\langle s\rangle$ with respect to $\langle o\rangle$. If we could find a rotation angle ϕ about $\boldsymbol{n}$ such that $\theta(\phi) = 0$ (after the preliminary maneuver), then the uniqueness of $\boldsymbol{v}_0$ would be violated because by assumption $\boldsymbol{n}^T\boldsymbol{v}_0 \in (-1, 1)$. □

Remark: Condition $\boldsymbol{n}^T\boldsymbol{v}_0 \neq \pm 1$ reported in the preceding lemma can be easily justified using trivial geometric considerations. If $\boldsymbol{n}$ and $\boldsymbol{v}_0$ were perfectly aligned, the meridians of $\boldsymbol{n}$ and those of $\boldsymbol{v}_0$ could intersect each other either only in the two poles for $\phi \neq \theta_0$ or would be the same circles for $\phi = \theta_0$.

From Eq. (75) this means that if $\phi \neq \theta_0$, that is, we rotate along the equivalent angle axis of an angle different from that required by Eq. (3), unit vector $\boldsymbol{v}$ of the equivalent axis would not change. On the other hand, if $\phi = \theta_0$, we have rotated of the exact angle required by Eq. (3) and, hence, at the end of the maneuvering θ would be zero and then $\boldsymbol{v}$ could not be uniquely defined [in this case Eq. (75) would be an identity]. This is actually coherent with the unconstrained case, where $\boldsymbol{\omega}$ was chosen along $\boldsymbol{v}_0$ and there was no change in $\boldsymbol{v}$, whereas only θ was affected by $\boldsymbol{\omega}$. □

The result of the preceding lemma allows us to state the following.

Corollary: In the constrained case we can implement the preliminary maneuvering choosing any $\boldsymbol{n}$ lying on the x–y plane. Moreover, there exists a rotation $\phi^*(\rho)$ such that

$$\gamma(\theta^*) = \min\left\{\frac{\Omega_m\theta(\phi^*)\cos\alpha(\phi^*)}{K\alpha(\phi^*)}, \gamma^*\right\} = \rho\gamma^*, \qquad \rho \in (0, 1) \tag{77}$$

Proof: The first statement is obvious. The second statement can be straightforwardly proven noting that, for continuous angular velocities directed along $\boldsymbol{n}$, all of the descriptive variables are continuous and

$$\lim_{\phi\to\phi^0} \frac{\Omega_m\theta(\phi)\cos\alpha(\phi)}{K\alpha(\phi)} = \infty \tag{78}$$

□

The rationale for a control law capable of preliminarily making (if the case) the gain γ, given by Eq. (62), greater than $\rho\gamma^*$ then follows directly: It simply consists of a continuous increment of angle ϕ starting from zero (and not exceeding the 2π value) till the fulfillment of the equality condition reported within Eq. (71). Obviously, whenever needed, the preceding objective can be easily obtained by applying, for instance, a positive constant angular velocity along $\boldsymbol{n}$, to be suddenly switched off at the time instant when threshold $\rho\gamma^*$ is reached. Alternatively, the following closed-loop control law (of asymptotic nature) can actually be used:

$$\omega = n\lambda\min\{S; \rho\gamma^* - \gamma\}, \qquad \lambda > 0, \quad S > 0 \tag{79}$$

with S a suitable threshold value.

Conclusions

The paper has considered the problem of asymptotically driving the attitude of a spacecraft via feedback control. It has been shown that the attitude parameterization known as equivalent angle axis can be effectively used to design simple feedback attitude control laws. These control strategies can be applied either when the system is fully actuated or when the spacecraft operates in a failure mode. An analysis concerning the induced angular accelerations has been carried out to ensure the boundedness of the demanded control signal in view of a possible application of the proposed scheme for the actual dynamic control of the real system.

The outcome of the analysis has shown that, in case of failure mode, a very simple sequence of two control actions can be defined to solve the reorientation control problem with bounded control signals. The first control, to be applied only if specific critical initial configurations of the spacecraft are met, operates for a finite time interval. This control, which may not reduce the actual orientation error, applies an angular velocity about a fixed and almost arbitrary rotation axis until suitable geometric conditions are fulfilled. Beyond this point, the proposed smooth control law, which ensures the asymptotic convergence to zero of the orientation errors, can be safely applied.

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